Inverse Heat Conduction Problem: Filter Solution With Given Boundary Temperature History

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ADVANTAGES OF FILTER APPROACH

- "EXPERT" INVERSE ASPECTS SEPARATED FROM THE APPLICATIONS ASPECTS
- RELATIVELY SIMPLE IMPLEMENTATION IN MANUFACTURING SETTING - CONTINUOUS
- GOOD FOR REPETITIVE USES
- POTENTIAL FOR INSTRUMENTS
- INTRINSIC VERIFICATION ASPECTS

LITERATURE REVIEW

1981 DIEGO MURIO PUBLISHED PAPERS GIVING 4 KINDS OF INVERSE KERNELS

1985 BECK, BLACKWELL & ST. CLAIR BOOK ON IHCP

NEITHER GAVE REAL-TIME POTENTIAL OR APPLICATION TO MILDLY VARYING THERMAL PROPERTIES

2005 BECK DISCUSSED FILTER METHOD FOR QUENCHING OF SPHERES. *T*-DEPENDENT THERMAL PROPERTIES. SINGLE TEMPERATURE MEASUREMENT AT CENTER POINT (WHICH IS “INSULATED”)

NOW EXTEND TO *T* HISTORY GIVEN AT “KNOWN” SURFACE

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OUTLINE

- **WHAT IS THE INVERSE HEAT CONDUCTION PROBLEM (IHCP)?**

  PROBLEM WITH GIVEN *T*(*t*) AT *x = L*
- **SENSITIVITY COEFFICIENTS**
- **INVERSE SOLUTION USING WHOLE DOMAIN TIKHONOV REGULARIZATION, HAT FUNCTIONS**
- **FILTER CONCEPTS WITH GIVEN *T*(*t*) AT *x = L***
- **EXAMPLE**
- **CONCLUSIONS**

JVB1

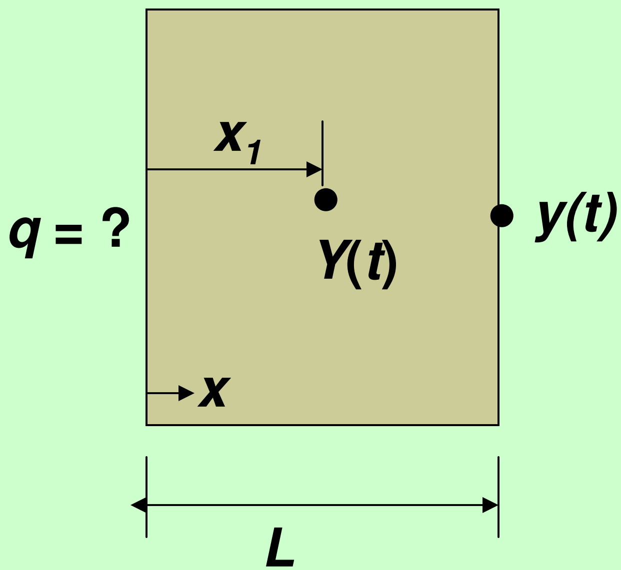
WHAT IS THE INVERSE HEAT CONDUCTION PROB.?

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x < L, \ \alpha = \text{thermal diffusivity = constant}$$

$$-k \frac{\partial T}{\partial x}(0,t) = q(t) = ?, \quad T(L,t) = y(t)$$

$$T(x,0) = T_0 = 0, \quad k = \text{thermal conductivity = constant}$$

$$T(x_1,t) = Y(t), \quad \text{given measured temp. at } x_1 = L/2$$



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JVB1 James Beck, 7/8/2006

TEMPERATURE AT x_1 SUM OF q AT $x = 0$ AND TEMPERATURE HISTORY AT $x = L$.

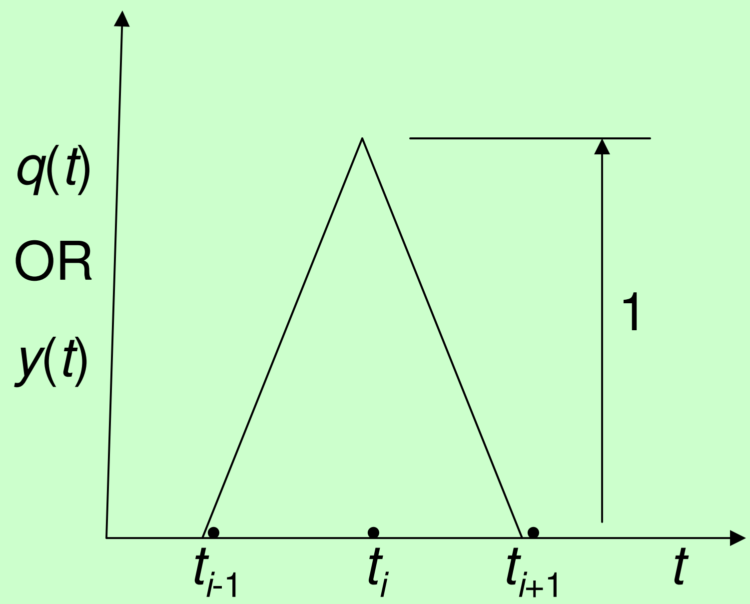
CAN BE DESCRIBED BY GREEN'S FUNCTIONS,

$$T(x_1,t)=\frac{\alpha}{k}\int_{\tau=0}^{t} q(\tau)G_{X21}(x_1,0,t-\tau)d\tau+\alpha\int_{\tau=0}^{t} y(\tau)\left(-\frac{\partial G_{X21}}{\partial x'}(x_1,L,t-\tau)\right)d\tau$$

$T(x_1,t)$ = "MEASURED" TEMP., ALSO DENOTED $Y(t)$
$q(t)$ = UNKNOWN HEAT FLUX TO BE ESTIMATED
$G(.)$ = GREEN'S FUNCTION
$y(t)$ = GIVEN TEMPERATURE HISTORY AT $x = L$

WE APROXIMATE USING "HAT" FUNCTIONS.

HAT BASIS FUNCTION FOR *q*(*t*) OR *y*(*t*)



$$t_i = i\Delta t$$

FOR LINEARLY INCREASING *q* AT *x* = 0,

$$\eta_i = \frac{L}{k}\frac{L^2}{\alpha\Delta t}\left\{\begin{array}{l}\left(1-\frac{x_1}{L}\right)\frac{\alpha i\Delta t}{L^2}-\frac{1}{6}\left[\left(\frac{x_1}{L}\right)^3-3\left(\frac{x_1}{L}\right)^2+2\right] \\ +2\sum_{m=1}^{\infty} e^{-\beta_m^2\frac{\alpha i\Delta t}{L^2}}\frac{\cos\left(\beta_m\frac{x_1}{L}\right)}{\beta_m^4}\end{array}\right\}$$

$\beta_m = (m\text{-}1/2)\pi$, FOR $x_1 = L/2$,

$$\eta_i = \frac{L}{k}\frac{L^2}{\alpha\Delta t}\left\{\frac{1}{2}\frac{\alpha i\Delta t}{L^2}-\frac{11}{48}+2\sum_{m=1}^{\infty} e^{-\beta_m^2\frac{\alpha i\Delta t}{L^2}}\frac{\cos\left(\frac{\beta_m}{2}\right)}{\beta_m^4}\right\}$$

INTRINSIC VERIFICATION POTENTIAL FOR αiΔt/L < 0.003

HAT FUNCTIONS (CONTINUED)

LINEARLY INCREASING *q* BOUNDARY CONDITION:

$$-k\frac{\partial \eta}{\partial x}(0,t_i) = \frac{t_i}{\Delta t} = i, \quad \eta \text{ units: K/(W/m}^2\text{)}$$

LET: $\eta_1 = T(L/2,\Delta t)\big|_{q_1=1}, \ \eta_2 = T(L/2,2\Delta t)\big|_{q_1=1,q_2=2}, \ \eta_i = T(L/2,i\Delta t)\big|_{q_1=1,q_2=2,\ldots,q_i=i}$

THEN FOR "HAT" BASIS FUNCTION

$$T_{q,M} = \sum_{i=1}^{M} q_i \delta^2 \eta_{M-i} = \sum_{i=1}^{M} q_i X_{M-i+1}$$

$$\delta^2 \eta_M = \eta_{M-1} - 2\eta_M + \eta_{M+1} = X_{M+1}$$

$$\partial T_{q,M} / \partial q_i = X_{M-i+1} = \text{SENS. COEF.}$$

SIMILARLY FOR THE *x = L* BOUNDARY CONDITION

$$\theta(L,t_i) = \frac{t_i}{\Delta t} = i, \quad \theta \text{ is dimensionless}$$

HAT FUNCTIONS (CONTINUED)

LET: $\theta_1 = T(L/2, \Delta t)\big|_{T_1=1}$, $\theta_2 = T(L/2, 2\Delta t)\big|_{T_1=1, T_2=2}$, $\theta_i = T(L/2, i\Delta t)\big|_{T_1=1, T_2=2, \ldots, T_i=i}$

THEN FOR "HAT" BASIS FUNCTION

$$T_{T,M} = \sum_{i=1}^{M} y_i \delta^2 \theta_{M-i} = \sum_{i=1}^{M} y_i Z_{M-i+1}$$

$$\delta^2 \theta_M = \theta_{M-1} - 2\theta_M + \theta_{M+1} = Z_{M+1}$$

THEN FOR BOTH CONTRIBUTIONS (*q* AT *x* = 0, *T* AT *x* = *L*)

$$T_M = \sum_{i=1}^{M} q_i X_{M-i+1} + \sum_{i=1}^{M} y_i Z_{M-i+1}$$

MATRIX FORM OF TEMP., LINEAR ANALYSIS

$\mathbf{T} = \mathbf{Xq} + \mathbf{Zy}$ **WHERE THE INITIAL TEMPERATURE = 0**

T = Xq

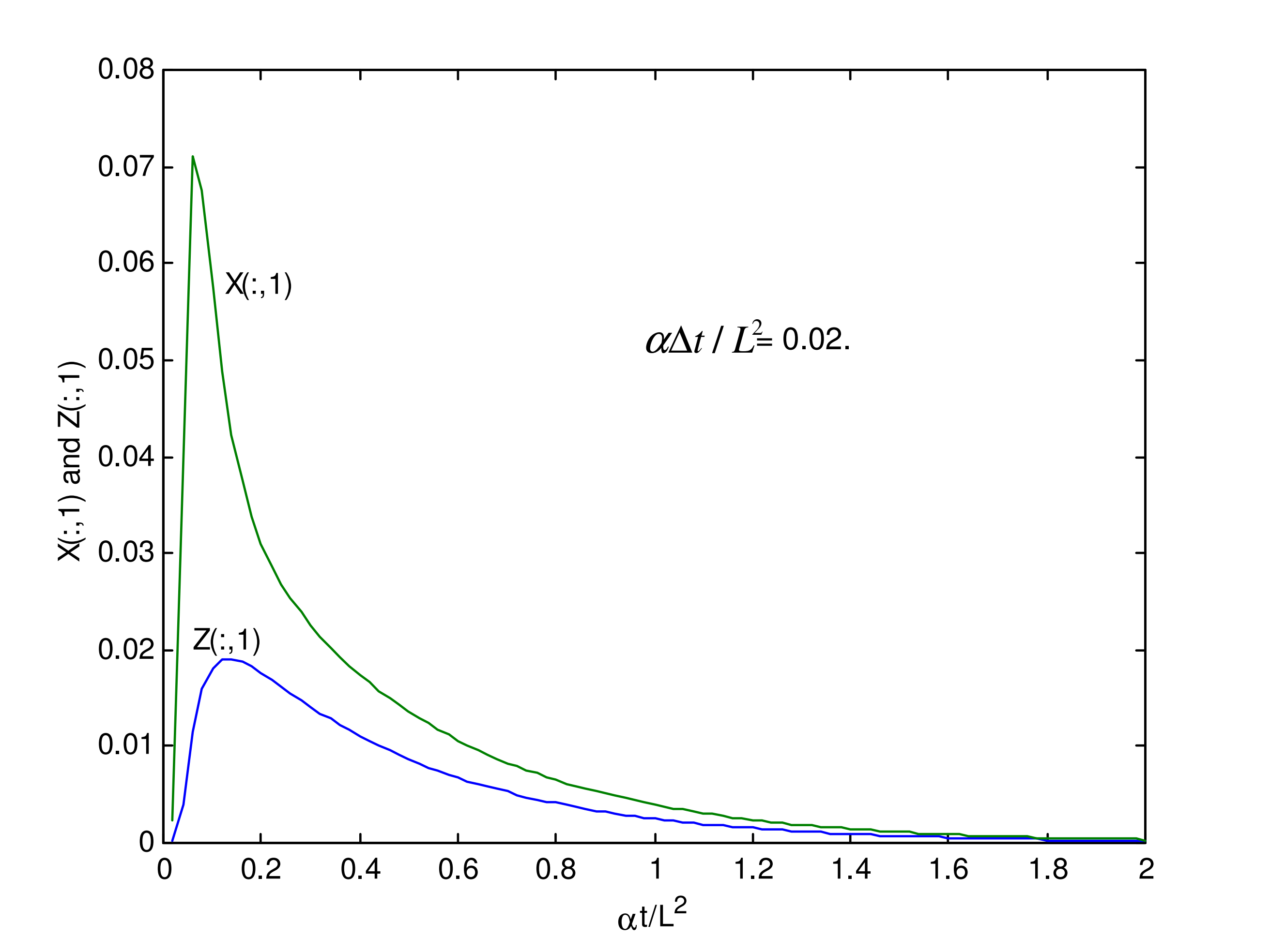
$$\begin{matrix} & & & & & \dfrac{\partial T}{\partial q_1} & \dfrac{\partial T}{\partial q_2} & \dfrac{\partial T}{\partial q_3} & & \dfrac{\partial T}{\partial q_{N-1}} & \dfrac{\partial T}{\partial q_N} \end{matrix}$$

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 & 0 \\ X_2 & X_1 & 0 & \cdots & 0 & 0 \\ X_3 & X_2 & X_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ X_N & X_{N-1} & X_{N-2} & \cdots & X_2 & X_1 \end{bmatrix}$$

Hat basis function: $X_i = \delta^2 \eta_{i-1}$

SIMILAR MATRICES FOR Z AND y.

SENSITIVITY COEFFICIENTS, HAT BASIS FUNCTION



X(:,1) for hat *q* at x = 0 and *Z*(:,1) for hat *T* at *x* = *L*. *x* = *L*/2.

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1ST ORDER TIKHONOV REGULARIZATION

SUM OF SQUARES (WHOLE TIME DOMAIN)

$$S = (\mathbf{Y} - \mathbf{T})^T (\mathbf{Y} - \mathbf{T}) + \alpha_T \mathbf{q}^T \mathbf{H}^T \mathbf{H} \mathbf{X} \mathbf{q}$$

$$= (\mathbf{Y} - \mathbf{X}\mathbf{q} - \mathbf{Z}\mathbf{y})^T (\mathbf{Y} - \mathbf{X}\mathbf{q} - \mathbf{Z}\mathbf{y}) + \alpha_T \mathbf{q}^T \mathbf{H}^T \mathbf{H} \mathbf{q}$$

α_T IS TIKHONOV REGULARIZATION PARAMETER, H HAS -1 ON MAIN DIAGONAL AND 1 ON DIAGONAL JUST ABOVE.

MINIMIZING SUM OF SQUARES GIVES

$$\hat{\mathbf{q}} = [\mathbf{X}^T \mathbf{X} + \alpha_T \mathbf{H}^T \mathbf{H}]^{-1} (\mathbf{X}^T \mathbf{Y} - \mathbf{Z}^T \mathbf{y})$$

NOTICE: ESTIMATES LINEAR FUNCTIONS OF Y & y. SUGGESTS USE OF FILTER COEFFICIENTS

FILTER COEFFICIENTS - DERIVATION

TRANSIENT HEAT CONDUCTION USING CONVOLUTION INTEGRALS & FUTURE INFO.

$$\hat{q}_M = \sum_{j=1}^{m_p + m_f} \left(f_j Y_{m_f + M - j} + g_j y_{m_f + M - j} \right)$$

$\hat{q}_M$ **is heat flux at time** t_M

f_j **and** g_i **are filter coefficients**

m_p **is for the past time steps**

m_f **is for the future time steps**

THE EQUATION IS THE BASIC ONE FOR FILTER ALGORITHM.
FOR GIVEN PROBLEM, f_J **AND** g_J **FOUND ONCE FOR ALL**

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HOW ARE f_j AND g_j FOUND?

$$\hat{q}_M = \sum_{j=1}^{m_p+m_f} f_j Y_{m_f+M-j} + \sum_{j=1}^{m_p+m_f} g_j y_{m_f+M-j}$$

$$= f_1 Y_{m_f+M-1} + f_2 Y_{m_f+M-2} + \cdots + f_{m_p+m_f} Y_{M-m_p} + g_1 y_{m_f+M-1} + \cdots + g_{m_p+m_f} y_{M-m_p}$$

LET $Y_j = y_j = 0 \text{ for all } j \text{ except } Y_{m_f} = 1$

USE LINEAR IHCP ALGORITHM (Funct. Spec., TIKHONOV REG.,..) TO GET ALL q_i ESTIMATES

SET M = 1: $\hat{q}_1 = f_1 Y_{m_f} + 0 = f_1 \cdot 1 = f_1,\ f_1 = \hat{q}_1$

SET M = 2: $\hat{q}_2 = f_2 Y_{m_f} + 0 = f_2 \cdot 1 = f_2,\ f_2 = \hat{q}_2$

SET j = M : $\hat{q}_M = \sum_{j=1}^{m_p+m_f} f_j Y_{m_f+M-j} = f_M Y_{m_f} = f_M, \quad f_M = \hat{q}_M$

FILTER COEFFICIENTS (CONTINUED)

LET $Y_j = y_j = 0 \text{ for all } j \text{ except } y_{m_f} = 1$

SET *M* = 1: $\hat{q}_1 = g_1 y_{m_f} + 0 = g_1 \cdot 1 = g_1, \;\; g_1 = \hat{q}_1$

SET *M* = 2: $\hat{q}_2 = g_2 y_{m_f} + 0 = g_2 \cdot 1 = g_2, \;\; g_2 = \hat{q}_2$

SET *j* = *M* : $\hat{q}_M = \sum_{j=1}^{m_p+m_f} g_j y_{m_f+M-j} = g_M y_{m_f} = g_M, \quad g_M = \hat{q}_M$

UNITS: $\mathbf{T=Xq+Zy}$

T **in K or C,** ***y*** **in same units, K or C.** ***q*** **in W/m^2,**
X **in K/(W/m^2),** ***Z*** **is dimensionless.**
f **and** ***g*** **have same units of W/m^2-K.**

EXAMPLE: PLATE $0<x<L$, $T(L,t) = y(t)$, $T(L/2,t) = Y(t)$

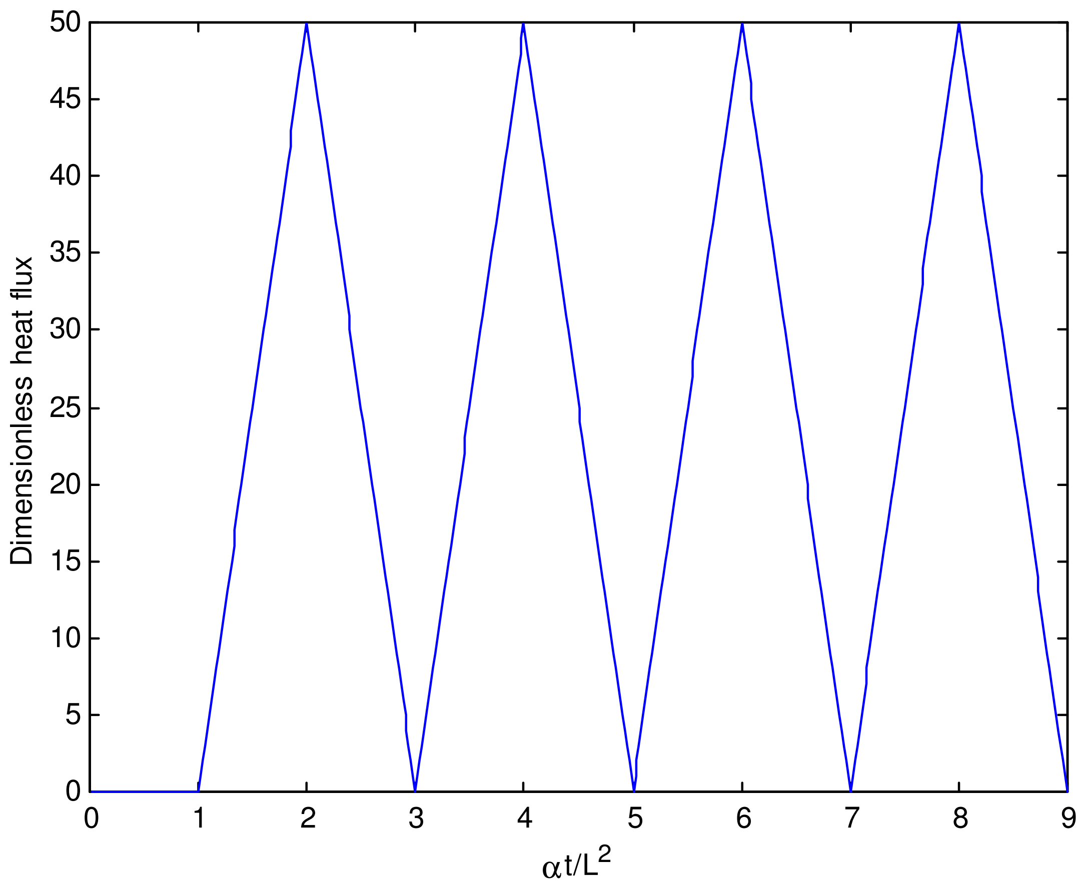
DIMENSIONLESS CASE: L **= 1 m,** k **= 1 W/m-K,**
α **= 1 m²/s**

WE CONSIDER A HEAT FLUX FORMED BY A SERIES OF HAT (TRIANGLES) FUNCTIONS, EACH $\alpha t/L^2 = 2$ **LONG.**

CALCULATE THE TEMPERATURE HISTORY USING EXACT SOLUTION FOR LENGTH = 2*L*.

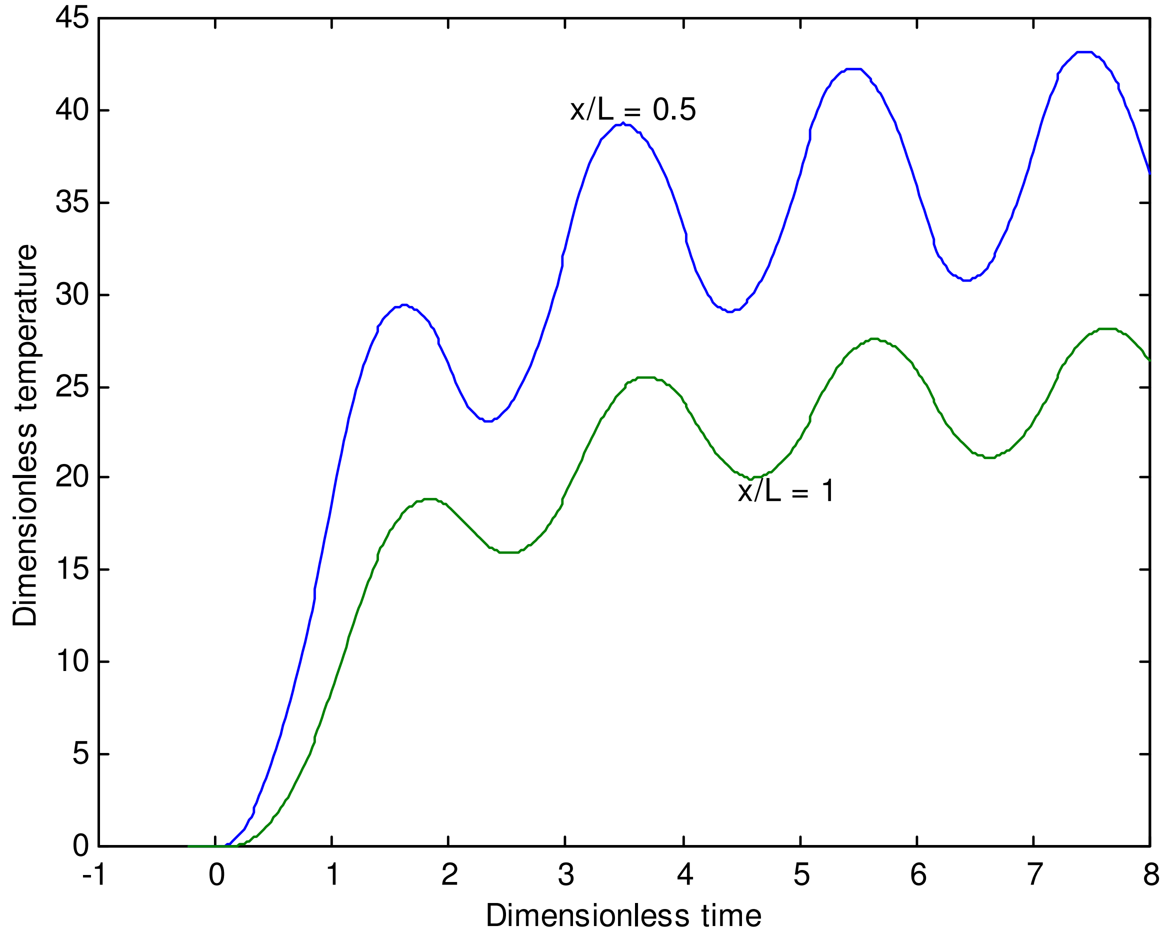
SIMULATED TIME STEPS, $\alpha \Delta t/L^2 = 0.02$

HEAT FLUX HISTORY FOR EXAMPLE



Hatplot.m x

TEMPERATURE HISTORIES IN PLATE



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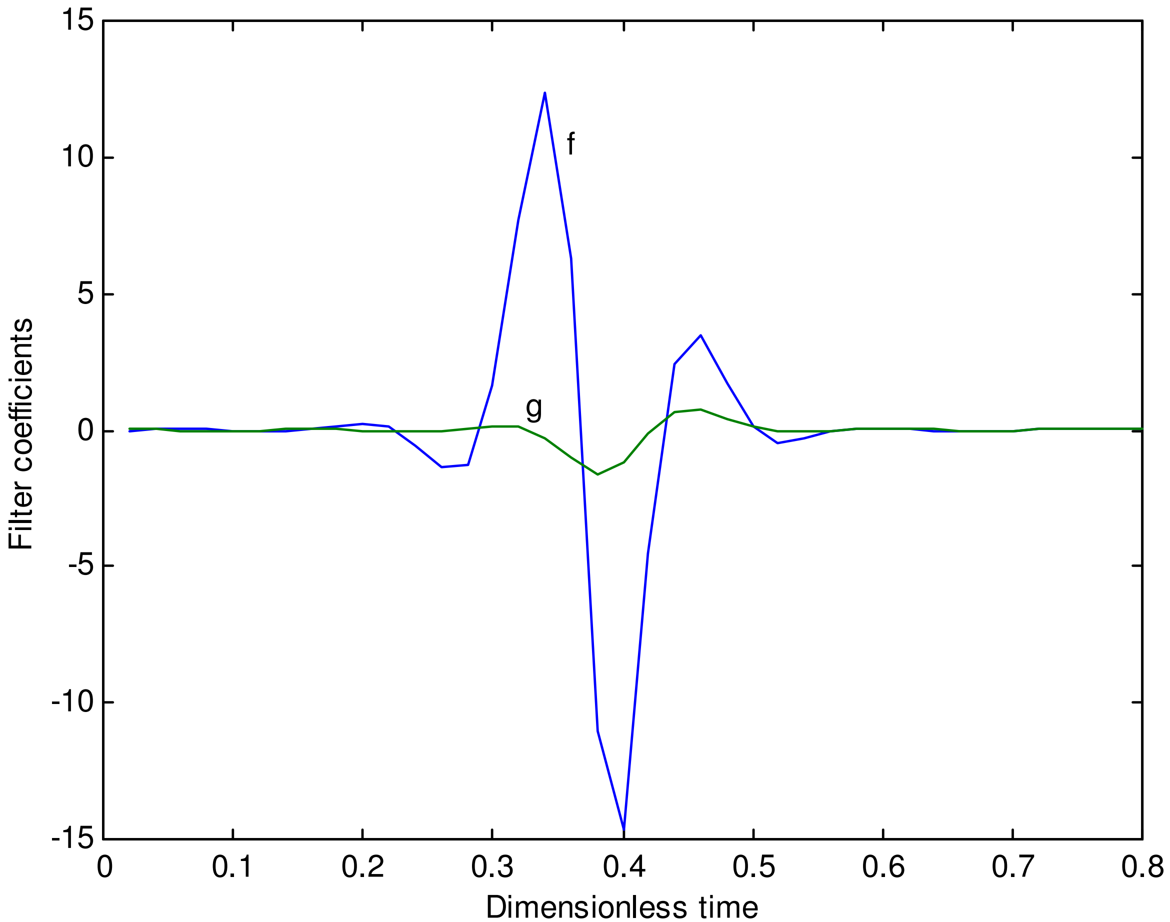
TIKHONOV IHCP FILTER COEFS., $\alpha_T = 0.0001$



SUM(*f*)= 1.9997, SUM(*g*) = -1.9999, $x_1 = L/2 = ½$.

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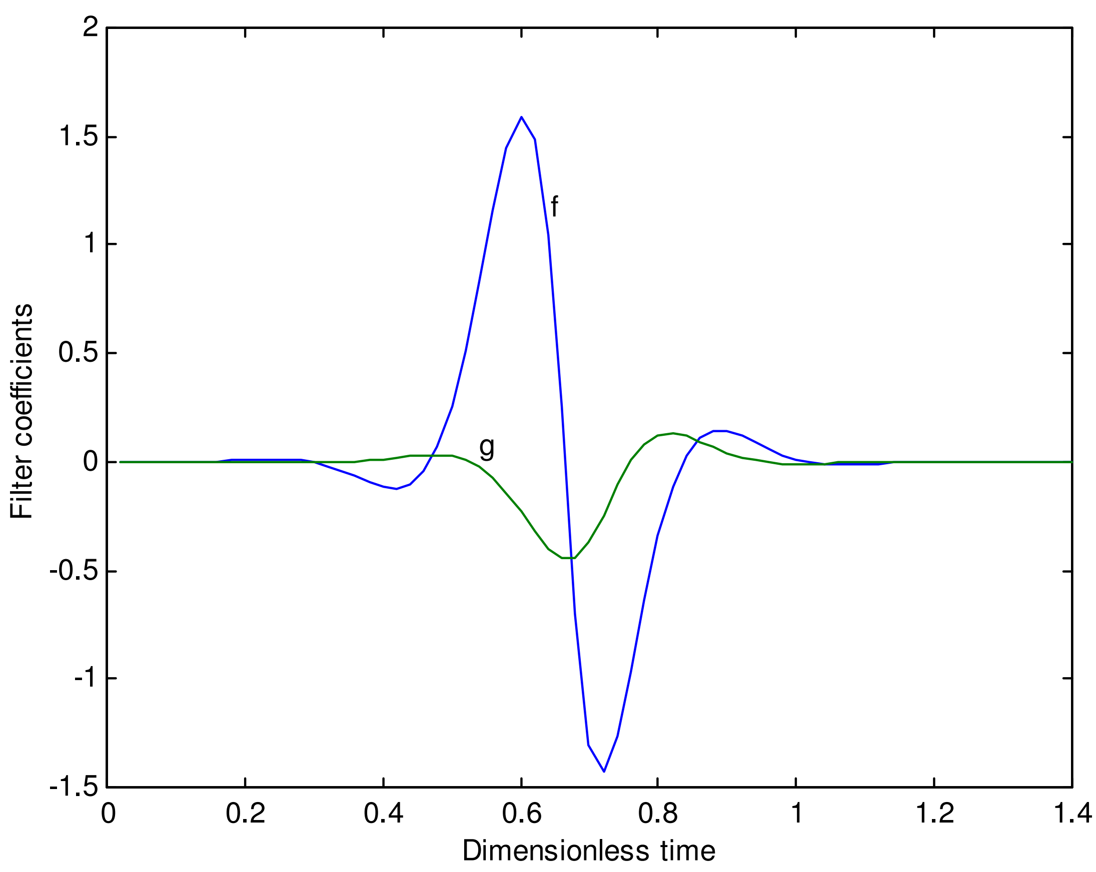
TIKHONOV IHCP FILTER COEFS., $\alpha_T = 0.01$



SUM(*f*)= 2.0024, SUM(g) = -1.9999, x_1 = *L*/2 = ½.

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INTRINSIC VERIFICATION USING SUMS

CAN THE SUM OF THE *f* AND *g* TERMS BE ANALYTICALLY DETERMINED?

STEADY-STATE PROBLEMS SOLVED WITH FILTER EQUATION.

FOR STEADY-STATE, $Y_j = Y_{SS}$, $y_j = y_{SS}$

$$\hat{q}_M = \sum_{j=1}^{m_p+m_f} \left(f_j Y_{m_f+M-j} + g_j y_{m_f+M-j} \right) = Y_{SS} \sum_{j=1}^{m_p+m_f} f_j + y_{SS} \sum_{j=1}^{m_p+m_f} g_j$$

ALSO STEADY- STATE HEAT CONDUCTION EQ. GIVES

$$\hat{q}_M = k \frac{Y_{SS} - y_{SS}}{L - x} = Y_{SS} \sum_{j=1}^{m_p+m_f} f_j + y_{SS} \sum_{j=1}^{m_p+m_f} g_j$$

$$Y_{SS} \left[\frac{k}{L-x} - \sum_{j=1}^{m_p+m_f} f_j \right] - y_{SS} \left[\frac{k}{L-x} + \sum_{j=1}^{m_p+m_f} g_j \right] = 0$$

INTRINSIC VERIFICATION (CONTINUED)

SINCE Y_{SS} & y_{ss} ARE INDEPENDENT,

$$\sum_{j=1}^{m_p+m_f} f_j = \frac{k}{L-x}, \quad \sum_{j=1}^{m_p+m_f} g_j = -\frac{k}{L-x}$$

NOW k = 1, L = 1, x = ½, THEN

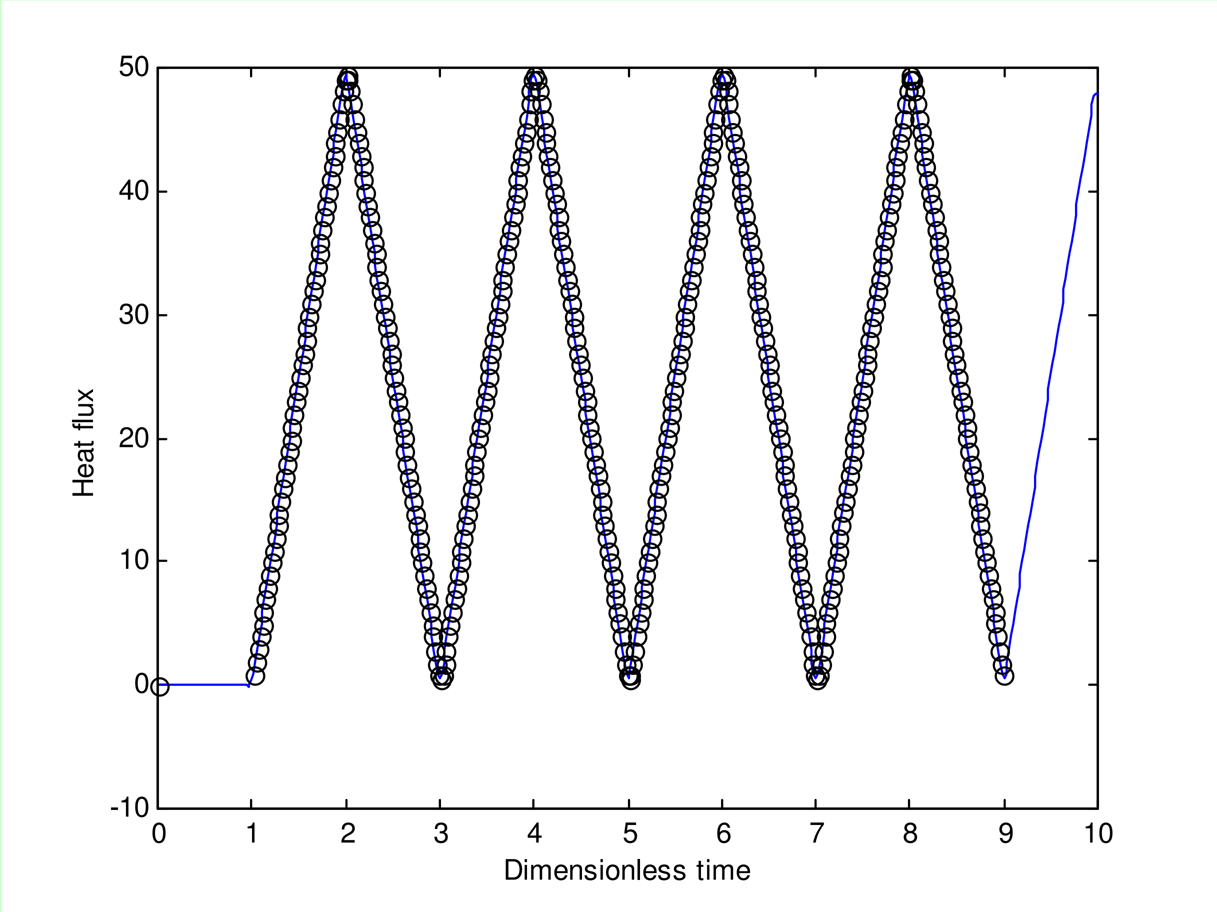
$$\sum_{j=1}^{m_p+m_f} f_j = 2, \quad \sum_{j=1}^{m_p+m_f} g_j = -2$$

NUMERICALLY EVALUATING THESE SUMS SHOULD GIVE ABOUT THE SAME VALUES.

FOR α_T = 0.0001, SUM(f) = 1.9997 AND SUM(g) = -1.9999.

HENCE, WE HAVE SHOWN INTRINSIC VERIFICATION.

COMPARISON OF WHOLE-DOMAIN & FILTER ANAL.



ERRORLESS DATA, α_T = 0.0001,
STD TIK = 0.0856, STD FILTER = 0.0859

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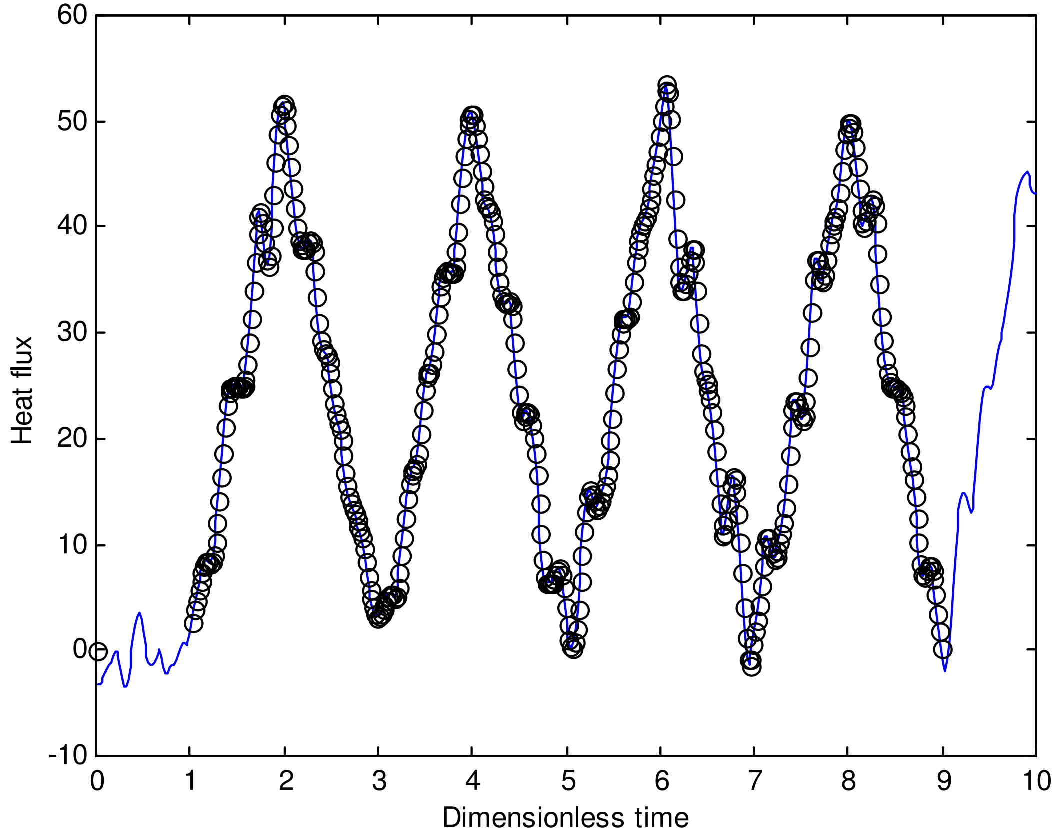
COMPARISON CONTINUED. DATA WITH ERRORS



ERRORS WITH STD. STATISTICAL ASSUMPTIONS,
σ = 0.5, α_T = 0.01, STD TIK = STD FILTER = 2.2814

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OBSERVATIONS

THE NUMERICAL VALUES ARE ALMOST IDENTICAL FOR THE WHOLE-DOMAIN ANALYSIS AND THE FILTER ANALYSIS.

COMPUTATIONALLY THE FILTER ANALYSIS IS MUCH MORE EFFICIENT.

APPLYING THE FILTER METHOD WITH KNOWN FILTER COEFFICIENTS IS MUCH EASIER THAN THE WHOLE-DOMAIN METHOD. MANY FEWER DECISIONS.

CAN MAKE AN INSTRUMENT TO DO THE FILTERING

CONCLUSIONS

METHOD FOR TREATING *T* BOUNDARY CONDITION

INITIAL CONDITION NOT NEEDED

WELL-SUITED FOR REPETITIVE TESTS OR CONTINUOUS USE

MAIN SKILL LEVEL NEEDED IN THE INVERSE ALGORITHM. LESS SKILL FOR FILTER SOL.

INTRINSIC VERIFICATION POSSIBLE